attention to the interpretation of low Reynolds number jet data.

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Elastic Beams of Various Orders

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General Theory

THE geometry, coordinate system, and some symbols are shown in Fig. 1. Plane state of stress in a homogeneous and isotropic beam of unit width subjected to loading

$$q = q_x(x)i + q_y(x)j \tag{1}$$

along $y = h$ is considered for establishing the general theory. The plane region A is bounded within $-L_1 \le x \le L_2$ and $0 \le y \le h$ by the boundary line S. Pertinent equations based on linear elasticity theory are listed below, and tensor notation is used for the convenience of presentation:

$$\iint \sigma_{ji,j}\delta u_i \mathrm{d}A + \int(\overset{n}{T_i} - \sigma_{ji}n_j)\delta u_i \mathrm{d}S = 0 \tag{2}$$

$$\sigma_{ij} = \frac{E}{1-\nu^2}\left[\nu e_{kk}\delta_{ij} + (1-\nu)e_{ij}\right] \tag{3}$$

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{4}$$

where σ_{ij} and e_{ij} for i and j ranging from 1 to 2 are the stress and strain tensors. $\overset{n}{T_i}$ is the surface traction at the boundary line with unit outward normal vector n_j. The modulus of elasticity, Poisson's ratio, shear modulus, and displacements are E, ν, G, and u_i, respectively. Equations (2-4) may be found in standard text books on mechanics of solids such as Refs. 1-3. It is clear that the first part of Eq. (2) involves the Euler's equations which are the equilibrium equations, and the second part contains the boundary conditions. The longitudinal displacement, $u_1 = u$, and the transverse displacement, $u_2 = w$ are represented by power series,

$$u = \sum_{m=0}^{\infty} U_m(x)y^m, \quad w = \sum_{m=0}^{\infty} W_m(x)y^m \tag{5}$$

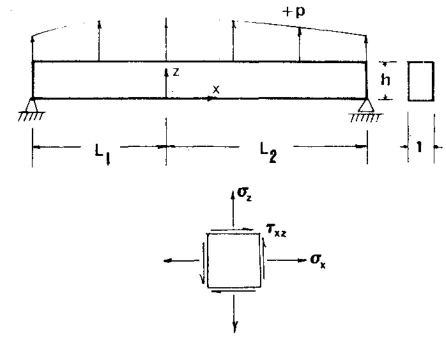

Fig. 1 Geometry, coordinates, and sign convention.

By substituting Eq. (5) into equilibrium equations contained in Eq. (2), and collecting like terms of y, one obtains the following recurrence relations:

$$U_{m+2} = -\frac{1}{(1-\nu)(m+2)}\left[(1+\nu)W'_{m+1} + \frac{2}{m+1}U''_m\right] \tag{6}$$

$$W_{m+2} = -\frac{1}{2(m+2)}\left[(1+\nu)U'_{m+1} + \frac{1-\nu}{m+1}W''_m\right] \tag{7}$$

where the prime denotes differentiation with respect to x. The free-of-stress boundary conditions along $y = 0$ require

$$U_1 = -W'_0, \quad W_1 = -\nu U'_0 \tag{8}$$

Consequently, all of the unknown coefficients for displacements shown in Eq. (5) can be expressed in terms of U_0 and W_0 and their derivatives. They may be written in the following general form:

$$\begin{Bmatrix} U_m \\ W_m \end{Bmatrix} = \begin{bmatrix} g_{mw} & g_{mu} \\ k_{mw} & k_{mu} \end{bmatrix} \frac{\mathrm{d}^m}{\mathrm{d}x^m}\begin{Bmatrix} W_0 \\ U_0 \end{Bmatrix} \tag{9}$$

in which $g_{mw} = k_{mu} = 0$ if m is an even integer, and $g_{mu} = k_{mw} = 0$ if m is an odd integer. Since g_{mw} and g_{mu} do not exist at the same time, a single symbol g_m will be used subsequently in place of g_{mw} and g_{mu}. Similarly, k_m will be used in place of k_{mw} and k_{mu}. The first few g_m and k_m are listed in Table 1. The stress components σ_x, τ_{xy} and σ_y can now be represented in power series of y with coefficients related to derivatives of U_0 and W_0,

$$\sigma_x = \sum_{i=0}^{I} \sigma_{xi}y^i = E\left(U'_0 - yW''_0 - y^2U'''_0 + \frac{1}{3}y^3W_0'^{v} + \ldots\right) \tag{10}$$

$$\tau_{xy} = \sum_{j=0}^{J} \tau_{xyj}y^j = E\left(-yU''_0 + \frac{1}{2}y^2W'''_0 + \frac{1}{3}y^3U_0'^{v} - \frac{1}{12}y^4W_0^{v} + \ldots\right) \tag{11}$$

$$\sigma_y = \sum_{k=0}^{K} \sigma_{yk}y^k = \frac{E}{2}\left(y^2U'''_0 - \frac{1}{3}y^3W_0'^{v} - \frac{1}{6}y^4U_0^{v} + \frac{1}{30}y^5W_0^{v\prime} + \ldots\right) \tag{12}$$

By satisfying the boundary conditions along $y = h$ contained in the second part of Eq. (2),

$$\tau_{yx} = q_x(x) \quad \text{and} \quad \sigma_y = q_y(x) \tag{13}$$

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Table 1 Displacement coefficients

m	g_m	k_m
0	1	1
1	-1	$-\nu$
2	$-(2+\nu)/2$	$\nu/2$
3	$(2+\nu)/6$	$(1+2\nu)/6$
4	$(3+2\nu)/24$	$-(1+2\nu)/24$
5	$-(3+2\nu)/120$	$-(2+3\nu)/120$

one obtains two coupled differential equations governing U_0 and W_0 when Eqs. (11) and (12) are used. The total order of the system of differential equations depends on the truncation of the series representing the stresses. The theory based on different order of series truncations for σ_x, τ_{xy}, and σ_y will be denoted as $I-J-K$ order theory reflected from Eqs. (10-12). If the equilibrium equations are to be exactly satisfied, $I=J-1=K-2$ must be followed. Otherwise, equilibrium equations are considered to be essentially satisfied, and the theory is referred to as an inconsistent theory. Inasmuch as inconsistent theories contain obvious flaws which have also shown in some numerical computations, only consistent theories will be presented in the study. Now the equilibrium condition for interior points and boundary conditions along $y=0$ and h are exactly satisfied, the remaining boundary conditions at $x=-L_1$ and L_2 contained in the second part of Eq. (2) are

$$\int_0^h (\sigma_x-\bar\sigma_x)(\delta U_0+g_1 y\delta W_0'+g_2 y^2\delta U_0''+\ldots)\mathrm{d}y=0 \tag{14}$$

$$\int_0^h (\tau_{xy}-\bar\tau_{xy})(\delta W_0+k_1 y\delta U_0'+k_2 y^2\delta W_0''+\ldots)\mathrm{d}y=0 \tag{15}$$

where $\bar\sigma_x$ and $\bar\tau_{xy}$ are prescribed stresses at boundary sections. Since the orders of series truncations for stresses have been set in Eqs. (10-12) with $I=J-1=K-2$ for consistent theories, the series representing displacements must be truncated in order to provide an adequate number of boundary conditions consistent with the total order of the system of governing differential equations. Furthermore, in order to maintain the same order of truncations in energy, one would truncate the series in Eq. (14) for u at one order higher than w in Eq. (15). As a result, a completely consistent beam theory will have an odd number of boundary conditions at each end. The conditions which may be prescribed at a boundary section of a beam are either N_i and V_j, or U_0 and W_0 and their derivatives as may be seen in Eqs. (14) and (15), where

$$N_i=\int_0^h y^i\sigma_x \mathrm{d}y \qquad i=0,1,2,3,\ldots,n$$

$$V_j=\int_0^h y^j\tau_{xy}\mathrm{d}y \qquad j=0,1,2,\ldots,n-1$$

These conditions together with governing differential equations obtained from Eq. (13) finalize the formulation of consistent beam theories.

1-2-3 and 3-4-5 Order Theories

For the 3-4-5 order theory, the differential equations are

$$U_0''-\frac{1}{2}hW_0'''-\underline{\frac{1}{3}h^2U_0^{iv}+\frac{1}{12}h^3W_0^{v}}=-\frac{1}{Eh}q_x \tag{16}$$

$$U_0'''-\frac{1}{3}hW_0^{iv}-\underline{\frac{1}{6}h^2U_0^{v}+\frac{1}{30}h^3W_0^{v\prime}}=\frac{2}{Eh^2}q_y \tag{17}$$

The quantities can be prescribed as boundary conditions are either:

$$N_0, N_1, V_0, \underline{N_2 \text{ and } V_1}$$

or:

$$U_0, W_0', W_0, \underline{U_0'' \text{ and } U_0'} \tag{18}$$

respectively. The displacements will be calculated from

$$u=U_0-yW_0'-\underline{\frac{2+\nu}{2}y^2U_0''},\quad w=W_0-\underline{\nu yU_0'} \tag{19}$$

The expressions for N_j and V_j are

$$N_j=g_j\sum_{m=0}^{3}\sigma_{xm}\frac{h^{m+j+1}}{m+j+1},\quad V_j=k_j\sum_{m=0}^{4}\tau_{xym}\frac{h^{m+j+1}}{m+j+1} \tag{20}$$

If the underlined quantites shown in Eqs. (16-19) are omitted, one arrives at the 1-2-3 order theory. It may be noted that the lowest 1-2-3 order theory is, in fact, the same as the elementary technical beam theory except that σ_y is generally ignored in the elementary theory. While general solutions for the 1-2-3 order theory can be easily obtained, the complete solution for W_0 based on 3-4-5 order theory is found to be

$$W_0=\sum_{i=1}^{4}B_iY_i(x)+\frac{h^4}{12}\left[A_0\left(x^3+\frac{4}{5}h^2x\right)+B_0\left(x^2+\frac{4h^2}{15}\right)+C_0x+D_0\right] \tag{21}$$

where B_i, A_0, B_0, C_0, and D_0 are integration constants. The functions Y_i are

$$Y_1=\cosh\alpha x\cos\beta x,\qquad Y_2=\sinh\alpha x\sin\beta x \tag{22a}$$

$$Y_3=\sinh\alpha x\cos\beta x,\qquad Y_4=\cosh\alpha x\sin\beta x \tag{22b}$$

and $\alpha h=2.42341$ and $\beta h=1.36856$. The complete solution for U_0 involving two additional integration constants, can be subsequently obtained from Eq. (16),

$$U_0=B_1(C_{11}Y_3+C_{21}Y_4)+B_2(C_{12}Y_3+C_{22}Y_4)+B_3(C_{11}Y_1+C_{21}Y_2)+B_4(C_{12}Y_1+C_{22}Y_2)+\frac{h^5}{24}\left[A_0\left(3x^2-\frac{9}{5}h^2\right)+2B_0x+C_0\right]+C_1x+D_1 \tag{23}$$

where

$$JC_{11}=a_{11}b_{11}+a_{12}b_{21},\quad JC_{12}=a_{11}b_{12}+a_{12}b_{22}$$

$$JC_{21}=a_{21}b_{11}+a_{22}b_{21},\quad JC_{22}=a_{21}b_{12}+a_{22}b_{22}$$

$$J=[1-⅓h^2(\alpha^2-\beta^2)]^2+(⅔h^2\alpha\beta)^2$$

$$a_{11}=a_{22}=1-⅓h^2(\alpha^2-\beta^2),$$

$$b_{11}=b_{22}=\frac{h}{2}\alpha\left[1-\frac{1}{6}h^2(\alpha^2-3\beta^2)\right]$$

$$a_{12}=-a_{21}=⅔h^2\alpha\beta,$$

$$b_{12}=-b_{21}=\frac{h}{2}\beta\left[1-\frac{1}{6}h^2(3\alpha^2-\beta^2)\right]$$

Table 2 σ_x^* at $x=0$

y/h	0	0.2	0.4	0.6	0.8	1.0
σ_x^*	1.0164	0.9897	0.9777	0.9743	0.9898	1.0179

Table 3 Stress ratio τ_{xy}^*

$y/h=$	0.1	0.3	0.5	0.7	0.9
$x=0.8L$	1.0156	1.0012	0.9950	0.9968	1.0066
$x=L$	0.9150	0.9265	0.9762	1.0640	1.1900

Table 4 Stress ratio σ_y^*

$y/h=$	0.1	0.3	0.5	0.7	0.9
$x=0.8L$	0.8397	0.8388	0.8719	0.9291	0.9875
$x=L$	0.0164	0.1425	0.3750	0.6737	0.9450

Table 5 W^* at $y=0$

x/L	0	0.2	0.4	0.6	0.8
W^*	1.1068	1.1071	1.1078	1.1086	1.1089

Table 6 Displacement ratio u^*

$x/L=$	0.2	0.4	0.6	0.8	1.0
$y=0$	1.0163	1.0160	1.0148	1.0122	1.0098
$y=h$	0.6034	0.5788	0.5343	0.4723	0.4281

Numerical Results and Discussions

For illustrative purposes, $q_x=0$, $q_y=p$, and $L_1=L_2=L$ are considered. The depth h is taken to be one-quarter of a unit. Rather short beams with $L=2h$ and $4h$, simply supported at $x=\pm L$ and $y=0$ are considered. The boundary conditions at $x=L$ for 1-2-3 and 3-4-5 order theories are $N_0=N_1=W_0=0$ and $N_0=N_1=N_2=W_0=U_0'=0$, respectively.

As a first example for which $p=P_0(L-x)$, Neou's[4] Airy polynomial stress function solution is comparable to the present 1-2-3 order theory. It is found that expressions for σ_y are identical, and discrepancies for σ_x and τ_{xy} are negligible between the two analyses. While the displacements can be calculated or be prescribed as boundary conditions in the present analysis, they can not be easily included in Ref. 4.

As a second example, $p=$constant is considered. Results based on 1-2-3 and 3-4-5 order theories will be discussed. As the discrepancies between these two theories increase as the beam length decreases, comparison of numerical results for the shorter beam $L=2h$ will be given. Longitudinal stresses of σ_x calculated according to the 3-4-5 order theory are slightly lower than those of 1-2-3 order theory for $0.2h \le y \le 0.8h$, and reversed in the remaining portion. Some results of σ_x^* at $x=0$, with the superscript * denoting the ratio of the quantity based on 3-4-5 order theory to that of 1-2-3 order theory, are listed in Table 2. Results for the shearing stress τ_{xz} calculated according to both theories agree very well in most of the interior part of the beam. Deviation begins at approximately $x=0.8L$. In this region near the edge, some results on the stress ratio τ_{xy}^* are given in Table 3. Results on σ_y agree well for most of the interior region. They begin to deviate at approximately $x=0.6L$ for the shorter beam. While σ_y does not vary along the x axis for 1-2-3 theory, it generally exhibits sharp stress gradients near the beam edge according to the 3-4-5 order theory. Some results on the stress ratio σ_y^* are listed in Table 4.

While w does not vary through the beam thickness for the 1-2-3 order theory, the variation is also very small according to 3-4-5 order theory. Results on the displacement ratio w^* along $y=0$ are given in Table 5.

Results on u based on 1-2-3 and 3-4-5 order theories agree well for the lower portion, but deviate from each other in the upper portion of the beam. Deviations become larger for sections closer to the edge. The discrepancies at $y=h$, quite significant for the shorter beam, are evident in the results in Table 6 on the displacement ratio u^*.

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Conservative Implicit Method for Shock Wave Calculations

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Nomenclature

A	= cross-sectional area of a node
e	= internal energy in a node
ℓ	= length of a node
m	= flow Mach number
M	= mass in a node
P	= pressure
u	= velocity defined by $(W/\rho/A)$
U	$= e + \frac{1}{2}Mu^2$; total energy in a node
V	= volume of node
W	= mass flow rate
γ	= ratio of specific heat
ρ	= density
Subscripts	
j,k,l,m	= values at positions shown in Fig. 1
0	= upstream value

Introduction

NUMERICAL techniques for calculations in which shock waves undergo considerable change from a state of initial steady propagation are of interest in many situations. For example, the interaction of a normal shock wave with a discontinuous area constriction results in reflected and

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